Invariant and Unbiased Tests for Error Variances for Ordered Alternatives in Fixed Effects Linear Models

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Abstract

In this article we study the problem of testing for equality of variances of k independent normal linear models, with a common regression parameter, against ordered alternatives. As a uniformly most powerful invariant test may not exist even for the case where $k = 2$, a locally best invariant test is derived. Although such tests are proven not to exist when $k \geq 3$, we propose a test procedure which, under certain conditions on the design matrices, will be unbiased.

Keywords and Phrases: Homoscedasticity, Locally Best Invariant Tests, Ordered Alternatives, Simple Order Restriction, Unbiased Tests.

1 Introduction

Consider k independent fixed effects linear models where the i^{th} model is given by:

$$Y_i = \mathbf{X}_i\beta + \epsilon_i, \quad i = 1, 2, \ldots, k. \tag{1}$$

In the above model Y_i is a $n_i \times 1$ response vector, $\mathbf{X}_i$ is a $n_i \times p$ matrix of known design constants, and the random errors ϵ_i are mutually independent and normally distributed

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with $E(\epsilon_i) = 0$ and $\mathrm{Var}(\epsilon_i) = \sigma_i^2 \mathbf{I}$. Stacking the k linear models one above the other we have

$$Y = \mathbf{X}\beta + \epsilon, \tag{2}$$

where $Y = [Y_1' : \cdots : Y_k']'$, $\mathbf{X} = [\mathbf{X}_1' : \cdots : \mathbf{X}_k']'$, and $\epsilon = [\epsilon_1' : \cdots : \epsilon_k']'$. We shall assume that each $\mathbf{X}_i$ is full rank and denote the total sample size by $N = \sum_{i=1}^{k} n_i$.

Models such as (1) are encountered in situations where an experimenter conducts k independent groups of experiments or when k independent labs conduct the same experiment. For instance, in an ongoing research, the Organization for Economic Co-operation and Development (OECD), Paris, France, is studying the effects of various chemicals on the uterine weights of rats. Nineteen different labs world-wide are participating in this study. Although the protocol of the study is identical, since not all labs had similar amount of experience in conducting these experiments, the OECD hypothesizes that the variability in uterine weights depends upon the level of experience of a lab. Motivated by OECD's uterotrophic assay and the discussion in Khosla et al. (1979), Rai and Rao (1984), and Rao et al. (1987), in this article we study the problem of testing for *simple order restriction*, i.e.,

$$H_0 : \sigma_1^2 = \cdots = \sigma_k^2, \quad H_1 : \sigma_1^2 \leq \sigma_2^2 \leq \cdots \leq \sigma_k^2 \text{ with at least one strict inequality.} \tag{3}$$

For the case $k = 2$, in Section 2 we derive the locally best invariant (LBI) test for the hypothesis (3). For $k \geq 3$ it is demonstrated that LBI tests do not exist for testing (3). A modification to the LBI test under simple order restriction is suggested for the case where $k \geq 3$. In Section 3 conditions are derived under which the proposed test procedure is also unbiased.

2 Invariant Tests

We begin this section with a motivating example. In this example we demonstrate that a uniformly most powerful invariant (UMPI) test does not exist even in a very special case of the general problem considered in this article. Consequently we shall explore locally best invariant tests.

Example 2.1 Suppose $X_i \sim^{iid} N(\mu, \sigma_1^2)$, $i = 1, 2, \ldots, n$ and $Y_i \sim^{iid} N(\mu, \sigma_2^2)$, $i = 1, 2, \ldots, n$. Further, suppose that all X_i are independent of all Y_i's. Let the sample means and sample variances of the two samples be denoted by $\bar{X}$, s_1^2 and $\bar{Y}$, s_2^2, respectively. Note that $W_1 = \frac{n-1}{\sigma_1^2} s_1^2 \sim \chi^2_{n-1}$, $W_2 = \frac{n-1}{\sigma_2^2} s_2^2 \sim \chi^2_{n-1}$ and $W_3 = \frac{n(\bar{X}-\bar{Y})^2}{\sigma_1^2+\sigma_2^2} \sim \chi^2_1$. The parameter of interest is $\rho = \sigma_2^2/\sigma_1^2$. Then the maximal invariant statistics, under the linear transformation group $x \to a + bx$, are

$$\begin{aligned}
(Z_1, Z_2) &= \left(\frac{n(\bar{X}-\bar{Y})^2}{(n-1)s_1^2}, \frac{s_2^2}{s_1^2} \right) \\
&= \left(\frac{n(\bar{X}-\bar{Y})^2}{\sigma_1^2+\sigma_2^2} \frac{\sigma_1^2}{(n-1)s_1^2} \frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2}, \frac{(n-1)s_2^2}{\sigma_2^2} \frac{\sigma_1^2}{(n-1)s_1^2} \frac{\sigma_2^2}{\sigma_1^2} \right) \\
&= \left((1+\rho)\frac{W_3}{W_1}, \rho \frac{W_2}{W_1} \right).
\end{aligned}$$

Since W_1, W_2 and W_3 are independently distributed, the joint density of W_1, W_2, W_3 is given by

$$f(w_1, w_2, w_3) = K w_1^{\frac{n-1}{2}-1} e^{-\frac{w_1}{2}} w_2^{\frac{n-1}{2}-1} e^{-\frac{w_2}{2}} w_3^{-\frac{1}{2}} e^{-\frac{w_3}{2}}.$$

Let

$$Z_1 = (1+\rho)\frac{W_3}{W_1}, \; Z_2 = \rho \frac{W_2}{W_1}, \text{ and } Z_3 = W_1.$$

Equivalently, we have

$$W_3 = \frac{Z_1 Z_3}{(1+\rho)}, \; W_2 = \frac{Z_2 Z_3}{\rho}, \text{ and } W_1 = Z_3.$$

The Jacobian of transformation is given by $Z_3^2/(\rho(1+\rho))$. Hence the joint distribution of (Z_1, Z_2, Z_3) is given by

$$\begin{aligned}
f(z_1, z_2, z_3) &= K z_3^{\frac{n-1}{2}-1} e^{-\frac{z_3}{2}} \left(\frac{z_2 z_3}{\rho} \right)^{\frac{n-1}{2}-1} e^{-\frac{z_2 z_3}{2\rho}} \left(\frac{z_1 z_3}{(1+\rho)} \right)^{-\frac{1}{2}} e^{-\frac{z_1 z_3}{2(1+\rho)}} \frac{z_3^2}{\rho(1+\rho)} \\
&= \frac{K}{\rho^{\frac{n-1}{2}} (1+\rho)^{\frac{1}{2}}} z_1^{-\frac{1}{2}} z_2^{\frac{n-1}{2}-1} z_3^{n-\frac{3}{2}} e^{-\frac{1}{2} z_3 (1 + \frac{z_2}{\rho} + \frac{z_1}{1+\rho})}.
\end{aligned}$$

Hence integrating z_3 we obtain the joint distribution of (Z_1, Z_2) as follows:

$$f(z_1, z_2) = \frac{K_1}{\rho^{\frac{n-1}{2}} (1+\rho)^{\frac{1}{2}}} \frac{z_1^{-\frac{1}{2}} z_2^{\frac{n-1}{2}-1}}{(1 + z_1/(1+\rho) + z_2/\rho)^{n-1/2}}.$$

Note that K and K_1 are some constants of integration.

Consider the problem of testing test $H_0 : \rho = 1$ against the simple hypothesis $H_1 : \rho = \rho_1$, where $\rho_1 > 1$. The likelihood ratio under the null and the alternative hypotheses reduces to

$$\frac{f_{H_0}}{f_{H_1}} = \frac{1}{\sqrt{2}} \left(\frac{z_1/(1+\rho_1) + z_2/\rho_1 + 1}{\frac{z_1}{2} + z_2 + 1} \right)^{n-\frac{1}{2}} \rho_1^{\frac{n-1}{2}} (1+\rho_1)^{\frac{1}{2}}.$$

Thus according to the Neyman Pearson Lemma the most powerful (invariant) test will reject the null hypothesis if

$$\left(\frac{z_1/(1+\rho_1) + z_2/\rho_1 + 1}{\frac{z_1}{2} + z_2 + 1} \right)^{n-\frac{1}{2}} \rho_1^{\frac{n-1}{2}} (1+\rho_1)^{\frac{1}{2}} \leq c, \tag{4}$$

where c is chosen so that the size of the test is α. The above test (4) is equivalent to

$$\left(\frac{z_1/(1+\rho_1) + z_2/\rho_1 + 1}{\frac{z_1}{2} + z_2 + 1} \right)^{n-\frac{1}{2}} \leq d, \tag{5}$$

where d is chosen so that the size of the test is α. Note that the above critical region (5) cannot be made independent of ρ_1. Consequently, the most powerful (invariant) test will depend upon the value of the parameter in the alternative space. Hence there is no uniformly most powerful invariant test for testing $H_0 : \rho = 1$ against $H_1 : \rho > 1$.

$\square$

Since the UMPI test fails to exist even for the simple case discussed in Example 2.1, we do not expect it to exist for the general problem considered in this article. Therefore in this section we shall discuss the existence of locally best invariant tests under the linear transformation group G defined by

$$g(Y) = c(Y + \mathbf{X}b), \ c > 0, \ \text{and} \ \ b \in \Re^p. \tag{6}$$

As in Mathew and Sinha (1988), we shall simplify the probability ratio of the maximal invariant under the alternative and null hypotheses using Wijsman's representation theorem (Wijsman, 1967). According to Wijsman's representation theorem the probability ratio may be written as

$$R = \frac{\int_G f_\delta\left(g\left(Y\right)\right) J_G^{-1} dG}{\int_G f_0\left(g\left(Y\right)\right) J_G^{-1} dG}. \tag{7}$$

In the above expression, J_G is the Jacobian, $dG = c^{-1}dbdc$ is the left invariant measure on G, $f_0(y)$ is the probability density function (pdf) of Y under H_0, and $f_\delta(y)$ is the pdf of Y under the alternative hypothesis.

We now describe some notations which will be used throughout this article.

Let $\|u\|$ indicate the euclidean norm of vector u, defined by $(u'u)^{1/2}$. Denote

$$p_i^2 = \frac{1}{n_i - p} \, ||Y_i - \mathbf{X}_i \left(\mathbf{X}'\mathbf{X}\right)^{-1} \mathbf{X}'Y||^2, \tag{8}$$

the mean squared residual based on an estimator of β which pools information from all of the groups, as if under homogeneity.

Let $\delta_i = \sigma_i^2 - \sigma_{i-1}^2$, $i = 2, \ldots, k$, and let $\delta = (\delta_2, \delta_3, \cdots, \delta_k)'$. Then the alternative hypothesis H_1 in (3) is equivalent to $\delta \geq 0$. We shall assume, without loss of generality, that $\sigma_1^2 = 1$. Observe that $\sigma_i^2 = 1 + \delta_2 + \cdots + \delta_i$ for $i = 2, \ldots, k$. Hence under the alternative hypothesis H_1, the covariance matrix of Y is a function of δ. Let this covariance matrix be denoted by $\mathbf{V}_\delta$, an $N \times N$ diagonal matrix. Under H_0 the covariance matrix of Y is an $N \times N$ identity matrix.

We now state the main result of this section. All proofs may be found in the Appendix.

Theorem 2.1 *For $k \geq 3$, locally best invariant tests do not exist for testing the hypotheses in (3). If $k = 2$, then the locally best invariant test of size α rejects the null hypothesis when $p_2^2/p_1^2 \geq c_\alpha$, where c_α is chosen such that $P\left(p_2^2/p_1^2 \geq c_\alpha | H_0\right) = \alpha$.*

The above theorem uses the following lemma which provides a Taylor's series expansion of R defined in (7).

Lemma 2.2 *Let $d_\delta = |\mathbf{X}'\mathbf{X}|^{(1/2)} / \left(|\mathbf{V}_\delta| \cdot |\mathbf{X}'\mathbf{V}_\delta^{-1}\mathbf{X}|\right)^{(1/2)}$ and $\theta = -(N-1)/2$. Let $\mathbf{0}$ denote an appropriately sized matrix of all zeros. Define $\mathbf{\Gamma}_i = [\mathbf{0} : \cdots : \mathbf{0} : \mathbf{I}_{n_i} : \mathbf{0} : \cdots : \mathbf{0}]'$ and*

$$\mathbf{\Lambda}_i = [\mathbf{0} : \cdots : \mathbf{0} : \mathbf{\Gamma}_i : \mathbf{\Gamma}_{i+1} : \cdots : \mathbf{\Gamma}_k]'.$$

Then

$$R = 1 + \left(\nabla d_\delta|_{\delta=0}\right)' \delta - \theta \left(\frac{\sum_{i=2}^k \delta_i \|\mathbf{\Lambda}_i' Y - \mathbf{\Lambda}_i' \mathbf{X} \left(\mathbf{X}'\mathbf{X}\right)^{-1} \mathbf{X}'Y\|^2}{\|Y - \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1} \mathbf{X}'Y\|^2} \right) + O\left(\delta'\delta\right). \tag{9}$$

The proof of Lemma 2.2 depends on the alternative hypothesis H_1 only through the structure of $\mathbf{V}_\delta$ and hence Λ_i. Thus the result is applicable to other order restrictions with appropriate modifications to the definition of Λ_i.

3 Unbiased Tests

Since no LBI test exists for $k \geq 3$, in this section we shall explore invariant tests which may at least be unbiased for testing against the simple order alternative given in (3). We begin with a simulation study.

We shall compare by computer simulation the powers of four different tests. The test procedures considered in this simulation study reject the null hypothesis for large values of the respective test statistic. The test statistics considered are (i) s_k^2/s_1^2, (ii) Hartley's statistic: $(\max_i s_i^2) \,/\, (\min_i s_i^2)$, (iii) p_k^2/p_1^2, and (iv) $(\max_i p_i^2) \,/\, (\min_i p_i^2)$. Test statistic (ii) is the well-known Hartley's F_{max} statistic for testing homogeneity of means. Test statistic (iii) is motivated by the LBI test statistic when $k = 2$. Test statistics (i) and (iv) are the regular and pooled analogs of (iii) and (ii), respectively.

We shall use the following three models for comparison purposes:

$$y_{ij} = 1 + \epsilon_{ij}, \quad i = 1, 2, \ldots, k, \; j = 1, 2, \ldots, n_i. \quad (CM)$$

$$y_{ij} = 1 + x_{ij} + \epsilon_{ij}, \quad i = 1, 2, \ldots, k, \; j = 1, 2, \ldots, n_i. \quad (SLR)$$

$$y_{ij} = 1 + x_{ij} + x_{ij}^2 + \epsilon_{ij}, \quad i = 1, 2, \ldots, k, \; j = 1, 2, \ldots, n_i. \quad (QUAD)$$

In each of the above models $\epsilon_{ij} \sim^{independent} N\left(0, \sigma_i^2\right)$. Clearly the CM and SLR models are special cases of the QUAD model. For purposes of the simulation, the number of groups k and the sample sizes within each group n_i were varied. For the SLR and QUAD models, the x_{ij} were set to $(i - 15)^j$.

In each case, the critical values for a nominal $\alpha = 0.05$ level test were based on 10,000 bootstrapped samples under the null hypothesis (without loss of generality, set to $\sigma_0^2 = \left(\sum \frac{\sigma_i}{k}\right)^2$). The power calculations were based on 10,000 bootstrapped samples in the alternative space where the variances were taken to be $\sigma_i^2 = i, \; i = 1, 2, \ldots, k$. Although it is well-known that statistic (i) follows an F distribution, for consistency all the power calculations were based on the bootstrapped critical values. It should be noted that the bootstrapped critical values for statistic (i) were close to the true F critical values. The resulting powers are summarized in Table 1.

Table 1: Power Comparisons for Ten Test Statistics

MODEL	k	n_i	N	(i)	(ii)	(iii)	(iv)
CM	6	2	12	.12	.06	.20	.07
CM	6	4	24	.37	.13	.43	.16
CM	6	6	36	.57	.24	.62	.29
CM	6	$2i$	42	.12	.10	.25	.14
CM	6	$3i$	63	.20	.15	.30	.20
CM	6	$4i$	84	.45	.33	.57	.37
CM	7	$4i$	112	.51	.40	.67	.45
CM	9	$4i$	180	.61	.48	.78	.59
CM	12	$4i$	312	.75	.63	.88	.74
SLR	6	4	24	.25	.09	.41	.16
SLR	6	6	36	.48	.18	.61	.30
SLR	6	$3i$	63	.10	.09	.30	.09
SLR	6	$4i$	84	.29	.21	.54	.22
SLR	7	$4i$	112	.30	.24	.63	.26
SLR	9	$4i$	180	.38	.30	.76	.36
SLR	12	$4i$	312	.43	.34	.87	.51
Q	6	4	24	.13	.05	.38	.16
Q	6	6	36	.36	.12	.58	.29
Q	6	$4i$	84	.12	.10	.52	.04
Q	7	$4i$	112	.13	.11	.63	.05
Q	9	$4i$	180	.15	.12	.74	.07
Q	12	$4i$	312	.17	.14	.87	.11

That Hartley's test does not perform well for the ordered alternative is not too surprising. Hartley's test was designed to detect hetergeneity in any form. When the alternative in (3) is true, we expect s_k^2 and s_1^2 to be within statistical error of $\max_i s_i^2$ and $\min_i s_i^2$, respectively. However, the critical values for Hartley's statistic are generally larger as $(\max_i s_i^2) / (\min_i s_i^2) \geq s_k^2/s_1^2$.

Clearly a simulation cannot be exhaustive in its cases. However, the variety of cases used suggests that pooling the information from other groups can dramatically improve the power of a test. Among these four statistics, (iii) consistently outperforms the others. As (iii) also provides an LBI test when $k = 2$, we now explore the unbiasedness property of a test based on (iii). In particular, we derive conditions under which this test can at least be unbiased for the simple order alternative, i.e. $P\left(p_k^2/p_1^2 \geq c_\alpha|H_1\right) \geq P\left(p_k^2/p_1^2 \geq c_\alpha|H_0\right)$.

Cohen et al. (1994) discuss the problem of testing for homogeneity of natural parameters against certain types of linear order restrictions when the sufficient statistics are independent and belong to single parameter exponential family with PF_2 property. Unfortunately, the results obtained in Cohen et al. (1994) are not applicable in the present context because the components of $(p_1^2, p_2^2, \ldots, p_k^2)$ are not independently distributed nor does the marginal distribution of p_i^2 belong to a single parameter exponential family with PF_2 property. Cohen and Sackrowitz (1993) derive some general stochastic inequalities which can be used for proving unbiasedness of certain types of test procedures. One of the assumptions made in Cohen and Sackrowitz (1993) requires the joint distribution of $(p_1^2, p_2^2, \ldots, p_k^2)$ to satisfy the decreasing in transposition (DT) property, which cannot be verified in the present context. For these reasons, we take a more direct approach to the problem.

Before we present the main result of this section, we define some additional notation. For any symmetric matrix $\mathbf{A}$, in this section we shall denote the set of eigenvalues by $\{\lambda_i(\mathbf{A})\}$. Here the index i does not correspond to the relative magnitude of the eigenvalue; rather it indicates that $\lambda_i(\mathbf{A})$ is the eigenvalue associated with the i^{th} eigenvector of $\mathbf{A}$, in whatever order those may be. Further, we shall denote $\mathbf{P}_{ij} = \mathbf{X}_i\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}_j'$ and

$$\mathbf{D}_{\delta i} = \begin{cases} \left[\mathbf{I}_{n_i} - \mathbf{P}_{ii}\right] + \sum_{t=2}^{k}\left[\delta_t \sum_{s=t}^{k} \mathbf{P}_{is}\mathbf{P}_{si}\right] & i = 1 \\ \left[\mathbf{I}_{n_i} - \mathbf{P}_{ii}\right] + \left(\sum_{t=2}^{i}\delta_t\right)\left[\mathbf{I}_{n_i} - 2\mathbf{P}_{ii}\right] + \sum_{t=2}^{k}\left[\delta_t \sum_{s=t}^{k} \mathbf{P}_{is}\mathbf{P}_{si}\right] & i = 2, \ldots, k \end{cases}.$$

Theorem 3.1 *If ϵ is multivariate normally distributed with mean 0 and covariance matrix $\Sigma = diag\left[\sigma_1^2\mathbf{I}_{n_1} : \cdots : \sigma_k^2\mathbf{I}_{n_k}\right]$ then a sufficient condition for*

$$\left.\frac{p_k^2}{p_1^2}\right|_\delta \geq_{stochastically} \left.\frac{p_k^2}{p_1^2}\right|_0, \tag{10}$$

is that

$$\lambda_j\left(\mathbf{D}_{01}\right)\frac{\partial\lambda_i\left(\mathbf{D}_{\delta k}\right)}{\partial\delta_t} \geq \lambda_i\left(\mathbf{D}_{0k}\right)\frac{\partial\lambda_j\left(\mathbf{D}_{\delta 1}\right)}{\partial\delta_t} \tag{11}$$

for all $i = 1, \ldots, n_k$; $j = 1, \ldots, n_1$; and $t = 2, \ldots, k$.

Corollary 3.2 *If $k = 2$, then (10) holds if (i) the eigenvalues of $\mathbf{P}_{11}$ are only 0's or 1's or (ii) all nonzero eigenvalues of $\mathbf{P}_{11}$ that are not equal to one are identical.*

We now present some experimental designs which satisfy the condition of Theorem 3.1.

Example 3.1 (Common Mean Model)

Here the model is

$$y_{ij} = \mu + \epsilon_{ij}, \quad i = 1, 2, \dots, k, \quad j = 1, 2, \dots, n_i, \tag{12}$$

where $\epsilon_{ij} \sim^{independent} N\left(0, \sigma_i^2\right)$. Let 1_{n_i} denote the $n_i \times 1$ vector of all ones and $\mathbf{J}_{n_i}$ denote the $n_i \times n_i$ matrix of all ones. Observe that the common mean model may be written in the form of (2) where $\mathbf{X} = 1_N$ and $\beta = \mu$. Note that in this case, $\left(\mathbf{X}'\mathbf{X}\right)^{-1} = 1/N$ and

$$\mathbf{D}_{\delta 1} \;\; = \;\; \mathbf{I}_{n_1} + \left[-(1/N) + \sum_{t=2}^{k} \left(\delta_t \sum_{s=t}^{k} n_s/N^2 \right) \right] \mathbf{J}_{n_1}.$$

Additionally

$$\begin{aligned} \mathbf{D}_{\delta k} \;\; = \;\; & \left(1 + \sum_{t=2}^{k} \delta_t\right) \mathbf{I}_{n_k} \\ & + \left[-\left(1 + 2\sum_{t=2}^{k} \delta_t\right)(1/N) + \sum_{t=2}^{k} \delta_t \left(n_k/N^2\right) + \sum_{t=2}^{k} \delta_t \left(\sum_{s=t}^{k-1} n_s/N^2 \right) \right] \mathbf{J}_{n_k}. \end{aligned}$$

Using the fact that the eigenvalues of $a\mathbf{I}_{n_i} + b\mathbf{J}_{n_i}$ are a with multiplicity $n_i - 1$ and $a + n_i b$ with multiplicity 1, the eigenvalues are determined and presented in Table 2 with the appropriate derivatives.

Observe that for $j = 1, \dots, n_1 - 1$, condition (11) from Theorem 3.1 is trivially satisfied as $\partial \lambda_j \left(\mathbf{D}_{\delta 1}\right) / \partial \delta_t = 0$ for all $t = 2, \dots, k$.

Consider the case for $j = n_1$. For $t = 2, \dots, k$ and $i = 1, \dots, n_k - 1$, condition (11) reduces to verifying

$$\left[\left(N - n_1\right)/N\right](1) \geq (1) \left(\sum_{s=t}^{k} n_1 n_s / N^2 \right)$$

or equivalently, $N\left(N - n_1\right) \geq n_1 \sum_{s=t}^{k} n_s$, which is true. For $t = 2, \dots, k$ and $i = n_k$ condition (11) simplifies to

$$\left[\left(N - n_1\right)/N\right] \left[\left(N - n_k\right)^2 / N^2 + \left(\sum_{s=t}^{k-1} n_k n_s / N^2 \right) \right] \geq \left[\left(N - n_k\right)/N\right] \left(\sum_{s=t}^{k} n_1 n_s / N^2 \right)$$

or equivalently,

$$\left(N - n_1\right) \left[\left(N - n_k\right)^2 + \left(n_k \sum_{s=t}^{k-1} n_s \right) \right] \geq \left(N - n_k\right) \left(n_1 \sum_{s=t}^{k} n_s \right).$$

Table 2: Eigenvalues and Derivatives for Common Mean Model

	$\lambda_j(\mathbf{D}_{\delta 1})$	$\partial\lambda_j(\mathbf{D}_{\delta 1})/\partial\delta_t$	$\lambda_j(\mathbf{D}_{01})$
$j = 1, \ldots, n_1 - 1$	1	0	1
$j = n_1$	$1 - (n_1/N)$ $+\sum_{t=2}^{k}\delta_t\left(\sum_{s=t}^{k} n_1 n_s/N^2\right)$	$\sum_{s=t}^{k} n_1 n_s/N^2$	$(N - n_1)/N$
	$\lambda_i(\mathbf{D}_{\delta k})$	$\partial\lambda_i(\mathbf{D}_{\delta k})/\partial\delta_t$	$\lambda_i(\mathbf{D}_{0k})$
$i = 1, \ldots, n_k - 1$	$1 + \sum_{t=2}^{k}\delta_t$	1	1
$i = n_k$	$\left(1 + \sum_{t=2}^{k}\delta_t\right)$ $-\left(1 + 2\sum_{t=2}^{k}\delta_t\right)(n_k/N)$ $+\sum_{t=2}^{k}\delta_t\left(n_k^2/N^2\right)$ $+\sum_{t=2}^{k}\delta_t\left(\sum_{s=t}^{k-1} n_k n_s/N^2\right)$	$(N - n_k)^2/N^2$ $+\left(\sum_{s=t}^{k-1} n_k n_s/N^2\right)$	$(N - n_k)/N$

Since $n_1\sum_{s=t}^{k} n_s \leq (N - n_k)(N - n_1)$, therefore

$$
\begin{aligned}
(N - n_k)\left(n_1\sum_{s=t}^{k} n_s\right) &\leq (N - n_k)^2(N - n_1) \\
&\leq (N - n_1)\left[(N - n_k)^2 + \left(n_k\sum_{s=t}^{k-1} n_s\right)\right].
\end{aligned}
$$

Hence the sufficient condition is satisfied.

Therefore, the test with rejection region $p_k^2/p_1^2 \geq c_\alpha$ is unbiased for the common mean model.

□

Example 3.2 (Replicated Linear Model)

In this case the linear model (1) is replicated k times. Thus $\mathbf{X}_i \equiv \mathbf{X}_1$, and $n_i \equiv n_1$, for $i = 1, 2, \ldots, k$. Replicated models have been well studied in the literature by many researchers. These models arise very naturally in fertilizer trials where the agronomist may want to study the repeatability of dose responses from year to year. Some useful references in this context are Rao et al. (1987), Rao et al. (1998), and Srivastava and Toutenburg (1994).

Notice that for the replicated model, $\mathbf{X}'\mathbf{X} = \sum_{i=1}^{k} \mathbf{X}_1'\mathbf{X}_1 = k\mathbf{X}_1'\mathbf{X}_1$ and so $(\mathbf{X}'\mathbf{X})^{-1}$ $= (1/k)\left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}$. As a result

$$\mathbf{D}_{\delta 1} = \mathbf{I}_{n_1} + \left[-(1/k) + \sum_{t=2}^{k} \delta_t (k-t+1)/k^2\right] \mathbf{X}_1 \left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1} \mathbf{X}_1'.$$

Additionally

$$\begin{aligned} \mathbf{D}_{\delta k} &= \left(1 + \sum_{t=2}^{k} \delta_t\right) \mathbf{I}_{n_1} \\ &\quad + \left[-\left(1 + 2\sum_{t=2}^{k} \delta_t\right)(1/k) + \sum_{t=2}^{k} \delta_t (k-t+1)/k^2\right] \mathbf{X}_1 \left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1} \mathbf{X}_1'. \end{aligned}$$

Note that as $\det\left[\lambda \mathbf{I}_{n_i} - (b\mathbf{I}_{n_i} + c\mathbf{A})\right] = \det\left[(\lambda - b)\mathbf{I}_{n_i} - c\mathbf{A}\right]$, the eigenvalues of $b\mathbf{I}_{n_i} + c\mathbf{A}$ are $b + c\lambda(\mathbf{A})$. Using this fact along with the fact that the eigenvalues of $\mathbf{X}_1\left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'$ are 1 with multiplicity $p = \text{Rank}(\mathbf{X}_1)$ and 0 with multiplicity $n_1 - p$, the eigenvalues are determined and presented in Table 3 with the appropriate derivatives.

Table 3: Eigenvalues and Derivatives for Replicated Linear Model

	$\lambda_j(\mathbf{D}_{\delta 1})$	$\partial\lambda_j(\mathbf{D}_{\delta 1})/\partial\delta_t$	$\lambda_j(\mathbf{D}_{01})$
$j = 1, \ldots, p$	$1 + \left[-(1/k) + \sum_{t=2}^{k} \delta_t (k-t+1)/k^2\right]$	$(k-t+1)/k^2$	$1 - (1/k)$
$j = p+1, \ldots, n_1$	1	0	1
	$\lambda_i(\mathbf{D}_{\delta k})$	$\partial\lambda_i(\mathbf{D}_{\delta k})/\partial\delta_t$	$\lambda_i(\mathbf{D}_{0k})$
$i = 1, \ldots, p$	$\left(1 + \sum_{t=2}^{k} \delta_t\right) + \left[-\left(1 + 2\sum_{t=2}^{k} \delta_t\right)(1/k) + \sum_{t=2}^{k} \delta_t (k-t+1)/k^2\right]$	$(1/k^2)(k^2 - k - t + 1)$	$1 - (1/k)$
$i = p+1, \ldots, n_1$	$\left(1 + \sum_{t=2}^{k} \delta_t\right)$	1	1

Observe that for $j = p+1, \ldots, n_1$, condition (11) from Theorem 3.1 is trivially satisfied as $\partial\lambda_j(\mathbf{D}_{\delta 1})/\partial\delta_t = 0$ for all $t = 2, \ldots, k$.

Consider the case for $j = 1, \ldots, p$. For $t = 2, \ldots, k$ and $i = p+1, \ldots, n_1$ condition (11) simplifies to verifying that

$$[1 - (1/k)](1) \geq (1)\left[(k-t+1)/k^2\right]$$

or equivalently that $k^2 - k \geq k - t + 1$. Since $k^2 - 2k + t - 1 = (k-1)^2 + t - 2$ and $t \geq 2$, the condition is met. Lastly, for $t = 2, \ldots, k$ and $i = 1, \ldots, p$ condition (11) reduces to

$$[1 - (1/k)] \left[\left(1/k^2 \right) \left(k^2 - k - t + 1 \right) \right] \geq [1 - (1/k)] \left[(k - t + 1) / k^2 \right]$$

or equivalently $k^2 - k - t + 1 \geq k - t + 1$. Since $k \geq 2$, this condition holds.

Therefore, the test with rejection region $p_k^2 / p_1^2 \geq c_\alpha$ is unbiased for the replicated model.

□

Appendix

We shall first provide the proof of Lemma 2.2 which is used in the proof of Theorem 2.1.

<u>Proof of Lemma 2.2</u>:

Without loss of generality we may assume that $\beta = 0$. Hence for $i = 0$ or δ,

$$f_i (g(Y)) = (2\pi)^{-(N/2)} |\mathbf{V}_i|^{-(1/2)} \exp \left\{ -\frac{1}{2} [c(Y + \mathbf{X}b)]' \mathbf{V}_i^{-1} [c(Y + \mathbf{X}b)] \right\}.$$

In the above, $\mathbf{V}_0$ indicates the covariance matrix of Y which is the $N \times N$ identity matrix.

Since the Jacobian of the transformation from Y to $g(Y)$ is c^{-N} and $dG = c^{-1} dbdc$, then

$$\begin{aligned} \int_G f_i (g(Y)) J_G^{-1} dG &= (2\pi)^{-(N/2)} |\mathbf{V}_i|^{-(1/2)} \\ &\quad \cdot \int_0^\infty \int_{\Re^p} \exp \left\{ -\frac{1}{2} c^2 (Y + \mathbf{X}b)' \mathbf{V}_i^{-1} (Y + \mathbf{X}b) \right\} c^{N-1} dbdc. \end{aligned}$$

To simplify notation let

$$Q_0 = Y'Y - Y'\mathbf{X} \left(\mathbf{X}'\mathbf{X} \right)^{-1} \mathbf{X}'Y,$$

and

$$Q_\delta = Y'\mathbf{V}_\delta^{-1} Y - Y'\mathbf{V}_\delta^{-1} \mathbf{X} \left(\mathbf{X}'\mathbf{V}_\delta^{-1} \mathbf{X} \right)^{-1} \mathbf{X}'\mathbf{V}_\delta^{-1} Y. \tag{13}$$

Note that

$$\begin{aligned} (Y + \mathbf{X}b)' \mathbf{V}_i^{-1} (Y + \mathbf{X}b) &= Y'\mathbf{V}_i^{-1} Y + 2b'\mathbf{X}'\mathbf{V}_i^{-1} Y + b'\mathbf{X}'\mathbf{V}_i^{-1} \mathbf{X}b \\ &= \left[b + \left(\mathbf{X}'\mathbf{V}_i^{-1} \mathbf{X} \right)^{-1} \mathbf{X}'\mathbf{V}_i^{-1} Y \right]' \mathbf{X}'\mathbf{V}_i^{-1} \mathbf{X} \\ &\quad \cdot \left[b + \left(\mathbf{X}'\mathbf{V}_i^{-1} \mathbf{X} \right)^{-1} \mathbf{X}'\mathbf{V}_i^{-1} Y \right] + Q_i. \end{aligned}$$

Thus

$$\begin{aligned}\int_G f_i(g(Y))\, J_G^{-1} dG &= (2\pi)^{-(N/2)} |\mathbf{V}_i|^{-(1/2)} \\ &\cdot \int_0^\infty \left[\int_{\Re^p} \exp\left\{ -\frac{c^2}{2} \left[b + \left(\mathbf{X}'\mathbf{V}_i^{-1}\mathbf{X}\right)^{-1} \mathbf{X}'\mathbf{V}_i^{-1} Y \right]' \mathbf{X}'\mathbf{V}_i^{-1}\mathbf{X} \right.\right. \\ &\cdot \left.\left. \left[b + \left(\mathbf{X}'\mathbf{V}_i^{-1}\mathbf{X}\right)^{-1} \mathbf{X}'\mathbf{V}_i^{-1} Y \right] \right\} db \right] c^{N-1} \exp\left\{ -\frac{c^2}{2} Q_i \right\} dc.\end{aligned}$$

Note that the inner integral is proportional to a p-variate normal density function with mean $-\left(\mathbf{X}'\mathbf{V}_i^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{V}_i^{-1}Y$ and covariance matrix $c^{-2}\left(\mathbf{X}'\mathbf{V}_i^{-1}\mathbf{X}\right)^{-1}$. It follows that

$$\int_G f_i(g(Y))\, J_G^{-1} dG = (2\pi)^{-[(N-p)/2]} |\mathbf{V}_i|^{-(1/2)} |\mathbf{X}'\mathbf{V}_i^{-1}\mathbf{X}|^{-(1/2)} \int_0^\infty c^{N-2} \exp\left\{ -\frac{c^2}{2} Q_i \right\} dc.$$

Under the change of variables $u = c^2 Q_i$ and $du = 2cQ_i dc$, then

$$\begin{aligned}\int_G f_i(g(Y))\, J_G^{-1} dG &= (1/2)\,(2\pi)^{-[(N-p)/2]} |\mathbf{V}_i|^{-(1/2)} |\mathbf{X}'\mathbf{V}_i^{-1}\mathbf{X}|^{-(1/2)} Q_i^{[-(N-1)/2]} \\ &\cdot \int_0^\infty u^{[(N-3)/2]} \exp\left\{ -\frac{u}{2} \right\} du.\end{aligned}$$

Thus

$$\begin{aligned}R &= \frac{(1/2)\,(2\pi)^{-[(N-p)/2]} |\mathbf{V}_\delta|^{-(1/2)} |\mathbf{X}'\mathbf{V}_\delta^{-1}\mathbf{X}|^{-(1/2)} Q_\delta^{[-(N-1)/2]} \int_0^\infty u^{[(N-3)/2]} \mathrm{e}^{-(u/2)} du}{(1/2)\,(2\pi)^{-[(N-p)/2]} |\mathbf{X}'\mathbf{X}|^{-(1/2)} Q_0^{[-(N-1)/2]} \int_0^\infty u^{[(N-3)/2]} \mathrm{e}^{-(u/2)} du} \\ &= d_\delta \left(\frac{Q_\delta}{Q_0} \right)^\theta .\end{aligned}$$

As $\delta_i \searrow 0$ for $i = 2, \ldots, k$, then $H_1 \to H_0$. Performing Taylor's series expansion of R about $\delta = 0$ we obtain

$$\begin{aligned}R &= d_0 \left(\frac{Q_0}{Q_0} \right)^\theta + \left[\left(\frac{Q_0}{Q_0} \right)^\theta \nabla d_\delta|_{\delta=0} + \left(\frac{d_0}{Q_0^\theta} \right) \theta Q_0^{(\theta-1)} \nabla Q_\delta|_{\delta=0} \right]' \delta + O(\delta'\delta) \\ &= 1 + (\nabla d_\delta|_{\delta=0})' \delta + \left(\frac{\theta}{Q_0} \right) (\nabla Q_\delta|_{\delta=0})' \delta + O(\delta'\delta) . \qquad (14)\end{aligned}$$

Note that

$$\begin{aligned}\frac{\partial Q_\delta}{\partial \delta_i} &= \left(\frac{\partial}{\partial \delta_i} Y'\mathbf{V}_\delta^{-1} Y \right) - \left(\frac{\partial}{\partial \delta_i} Y'\mathbf{V}_\delta^{-1}\mathbf{X} \right) \left(\mathbf{X}'\mathbf{V}_\delta^{-1}\mathbf{X}\right)^{-1} \mathbf{X}'\mathbf{V}_\delta^{-1} Y \\ &- Y'\mathbf{V}_\delta^{-1}\mathbf{X} \left[\frac{\partial}{\partial \delta_i} \left(\mathbf{X}'\mathbf{V}_\delta^{-1}\mathbf{X}\right)^{-1} \right] \mathbf{X}'\mathbf{V}_\delta^{-1} Y \\ &- Y'\mathbf{V}_\delta^{-1}\mathbf{X} \left(\mathbf{X}'\mathbf{V}_\delta^{-1}\mathbf{X}\right)^{-1} \left(\frac{\partial}{\partial \delta_i} \mathbf{X}'\mathbf{V}_\delta^{-1} Y \right).\end{aligned}$$

Since

$$\frac{\partial \mathbf{B}^{-1}}{\partial x} = -\mathbf{B}^{-1}\left(\frac{\partial \mathbf{B}}{\partial x}\right)\mathbf{B}^{-1},$$

therefore

$$\frac{\partial\left(\mathbf{A}'\mathbf{B}^{-1}\mathbf{A}\right)^{-1}}{\partial x} = \left(\mathbf{A}'\mathbf{B}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}'\mathbf{B}^{-1}\left(\frac{\partial \mathbf{B}}{\partial x}\right)\mathbf{B}^{-1}\mathbf{A}\left(\mathbf{A}'\mathbf{B}^{-1}\mathbf{A}\right)^{-1},$$

where $\mathbf{A}$ does not depend on x. Hence

$$\begin{aligned}
\frac{\partial Q_\delta}{\partial \delta_i} \;=\;& -Y'\mathbf{V}_\delta^{-1}\left(\frac{\partial \mathbf{V}_\delta}{\partial \delta_i}\right)\mathbf{V}_\delta^{-1}Y + Y'\mathbf{V}_\delta^{-1}\left(\frac{\partial \mathbf{V}_\delta}{\partial \delta_i}\right)\mathbf{V}_\delta^{-1}\mathbf{X}\left(\mathbf{X}'\mathbf{V}_\delta^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{V}_\delta^{-1}Y \\
& - Y'\mathbf{V}_\delta^{-1}\mathbf{X}\left(\mathbf{X}'\mathbf{V}_\delta^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{V}_\delta^{-1}\left(\frac{\partial \mathbf{V}_\delta}{\partial \delta_i}\right)\mathbf{V}_\delta^{-1}\mathbf{X}\left(\mathbf{X}'\mathbf{V}_\delta^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{V}_\delta^{-1}Y \\
& + Y'\mathbf{V}_\delta^{-1}\mathbf{X}\left(\mathbf{X}'\mathbf{V}_\delta^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{V}_\delta^{-1}\left(\frac{\partial \mathbf{V}_\delta}{\partial \delta_i}\right)\mathbf{V}_\delta^{-1}Y.
\end{aligned}$$

Since

$$\mathbf{V}_\delta = \mathrm{diag}\left[\mathbf{I}_{n_1} : (1+\delta_2)\,\mathbf{I}_{n_2} : \cdots : (1+\delta_2+\cdots+\delta_k)\,\mathbf{I}_{n_k}\right],$$

notice that

$$\left.\frac{\partial \mathbf{V}_\delta}{\partial \delta_i}\right|_{\delta=0} = \mathrm{diag}\left[\mathbf{0} : \cdots : \mathbf{0} : \mathbf{I}_{n_i} : \mathbf{I}_{n_{(i+1)}} : \cdots : \mathbf{I}_{n_k}\right] = \Lambda_i\boldsymbol{\Lambda}_i'.$$

It follows that

$$\begin{aligned}
\left.\frac{\partial Q_\delta}{\partial \delta_i}\right|_{\delta=0} \;=\;& -Y'\mathbf{V}_0^{-1}\Lambda_i\boldsymbol{\Lambda}_i'\mathbf{V}_0^{-1}Y + Y'\mathbf{V}_0^{-1}\Lambda_i\boldsymbol{\Lambda}_i'\mathbf{V}_0^{-1}\mathbf{X}\left(\mathbf{X}'\mathbf{V}_0^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{V}_0^{-1}Y \\
& - Y'\mathbf{V}_0^{-1}\mathbf{X}\left(\mathbf{X}'\mathbf{V}_0^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{V}_0^{-1}\Lambda_i\boldsymbol{\Lambda}_i'\mathbf{V}_0^{-1}\mathbf{X}\left(\mathbf{X}'\mathbf{V}_0^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{V}_0^{-1}Y \\
& + Y'\mathbf{V}_0^{-1}\mathbf{X}\left(\mathbf{X}'\mathbf{V}_0^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{V}_0^{-1}\Lambda_i\boldsymbol{\Lambda}_i'\mathbf{V}_0^{-1}Y \\
=\;& -\|\boldsymbol{\Lambda}_i'Y - \boldsymbol{\Lambda}_i'\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'Y\|^2,
\end{aligned}$$

since $\mathbf{V}_0 = \mathbf{I}_N$. Substituting the above into (14), then the lemma is proved.

□

<u>Proof of Theorem 2.1</u>:

Recall that $\Lambda_i = [\mathbf{0} : \cdots : \mathbf{0} : \boldsymbol{\Gamma}_i : \boldsymbol{\Gamma}_{i+1} : \cdots : \boldsymbol{\Gamma}_k]'$. It follows that $\boldsymbol{\Lambda}_i'Y = \sum_{j=i}^k \boldsymbol{\Gamma}_j Y_j$ and

$$\|\boldsymbol{\Lambda}_i'Y - \boldsymbol{\Lambda}_i'\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'Y\|^2 \;=\; \sum_{j=i}^{k}\|Y_j - \mathbf{X}_j\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'Y\|^2.$$

Let $SS_j = \|Y_j - \mathbf{X}_j (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'Y\|^2$. Thus the right hand side above is $SS_i + \cdots + SS_k$. Also note that

$$Q_0 = \sum_{i=1}^{k} \|Y_i - \mathbf{X}_i (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'Y\|^2 = SS_1 + SS_2 + \cdots + SS_k.$$

Hence, from Lemma 2.2 we have,

$$R = 1 + (\nabla d_\delta|_{\delta=0})' \delta + \left(\frac{N-1}{2}\right) \sum_{i=2}^{k} \delta_i \left(\frac{SS_i + \cdots + SS_k}{SS_1 + SS_2 + \cdots + SS_k}\right) + O(\delta'\delta).$$

Suppose $k = 2$ then $\delta = \delta_2$, a scalar. It follows that

$$\begin{aligned} R &= 1 + \delta \left(\left.\frac{\partial d_\delta}{\partial \delta}\right|_{\delta=0} \right) + \delta \left(\frac{N-1}{2}\right) \frac{SS_2}{SS_1 + SS_2} + O\left(\delta^2\right) \\ &= 1 + \delta \left(\left.\frac{\partial d_\delta}{\partial \delta}\right|_{\delta=0} \right) + \delta \left(\frac{N-1}{2}\right) \frac{1}{(SS_1/SS_2) + 1} + O\left(\delta^2\right). \end{aligned}$$

Note that $(\partial d_\delta / \partial \delta)|_{\delta=0}$ is free of random variables, $(N-1)/2 > 0$, and $\delta > 0$. Thus R is a monotonically increasing function of SS_2/SS_1. Therefore, the LBI test rejects H_0 when SS_2/SS_1 is large. Observe that $(n_i - p)\, p_i^2 = SS_i$ and $(n_i - p) > 0$. Thus $p_k^2/p_1^2 = [(n_1 - p) / (n_2 - p)] (SS_2/SS_1)$ is the LBI test when $k = 2$.

Suppose $k > 2$. Let $\delta_3 = \cdots = \delta_k = 0$. This is equivalent to choosing a point in the alternative space where $\sigma_1^2 < \sigma_2^2 = \sigma_3^2 = \cdots = \sigma_k^2$. Then

$$\begin{aligned} R &= 1 + (\nabla d_\delta|_{\delta=0})' \delta + \left(\frac{N-1}{2}\right) \delta_2 \left(\frac{SS_2 + \cdots + SS_k}{SS_1 + SS_2 + \cdots + SS_k}\right) + O(\delta'\delta) \\ &= 1 + (\nabla d_\delta|_{\delta=0})' \delta + \delta_2 \left(\frac{N-1}{2}\right) \frac{1}{[SS_1/(SS_2 + \cdots + SS_k)] + 1} + O\left(\delta_2^2\right). \end{aligned}$$

Thus the best invariant test at such a point in the alternative space would reject the null when $(SS_2 + \cdots + SS_k)/SS_1$ is large. On the other hand, suppose $\delta_2 = \cdots = \delta_{k-1} = 0$, which is equivalent to choosing a point in the alternative space where $\sigma_1^2 = \cdots = \sigma_{k-2}^2 = \sigma_{k-1}^2 < \sigma_k^2$. Then

$$\begin{aligned} R &= 1 + (\nabla d_\delta|_{\delta=0})' \delta + \left(\frac{N-1}{2}\right) \delta_k \left(\frac{SS_k}{SS_1 + SS_2 + \cdots + SS_k}\right) + O(\delta'\delta) \\ &= 1 + (\nabla d_\delta|_{\delta=0})' \delta + \delta_k \left(\frac{N-1}{2}\right) \frac{1}{[(SS_1 + \cdots + SS_{k-1})/SS_k] + 1} + O\left(\delta_k^2\right). \end{aligned}$$

Thus the best invariant test at such a point in the alternative space would reject the null when $SS_k/(SS_1 + \cdots + SS_{k-1})$ is large. Since the best invariant test depends on the point in the alternative space, no LBI test exists. Hence the theorem is proved.

$\square$

We need the following lemmas in the proof of Theorem 3.1.

Lemma A.3 *If ϵ is multivariate normally distributed with mean 0 and covariance matrix $\Sigma = diag\left[\sigma_1^2 \mathbf{I}_{n_1} : \cdots : \sigma_k^2 \mathbf{I}_{n_k}\right]$, then*

$$p_k^2/p_1^2 \overset{D}{=} \left(\frac{n_1 - p}{n_k - p}\right)\left(\frac{\sum_{i=1}^{n_k} \lambda_i\left(\mathbf{D}_{\delta k}\right)\mathbf{V}_{ki}}{\sum_{j=1}^{n_1} \lambda_j\left(\mathbf{D}_{\delta 1}\right)\mathbf{V}_{1j}}\right),$$

where $\mathbf{V}_{ij} \sim \chi_1^2$ such that $\mathbf{V}_{ij}$ is independent of $\mathbf{V}_{ij'}$, $j \neq j'$.

<u>Proof of Lemma A.3</u>: Let $Z_i = \mathbf{\Gamma}_i'\left(\mathbf{I}_N - \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\right)Y$, then

$$(n_i - p)\, p_i^2 = \|Y_i - \mathbf{X}_i\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'Y\|^2 = Z_i'Z_i.$$

Recall that if $u \sim N(0, \Sigma)$, then $u'u \overset{D}{=} \sum_{i=1}^{n_k} \lambda_i(\Sigma)\mathbf{V}_i$ where $\mathbf{V}_i \sim^{independent} \chi_1^2$. Note that $E\left(Z_i\right) = 0$, and

$$\begin{aligned}
\operatorname{Var}\left(Z_i\right) &= \mathbf{\Gamma}_i'\left[\mathbf{I}_N - \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\right]\operatorname{Var}(Y)\left[\mathbf{I}_N - \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\right]\mathbf{\Gamma}_i \\
&= \sigma_i^2\mathbf{I}_{n_i} - \sigma_i^2\mathbf{X}_i\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}_i' - \sigma_i^2\mathbf{X}_i\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}_i' \\
&\quad + \sum_{s=1}^{k}\sigma_s^2\mathbf{X}_i\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}_s'\mathbf{X}_s\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}_i'.
\end{aligned}$$

Since $\sigma_i^2 = 1 + \delta_2 + \cdots + \delta_i$, therefore

$$\begin{aligned}
\operatorname{Var}\left(Z_i\right) &= (1 + \delta_2 + \cdots + \delta_i)\left[\mathbf{I}_{n_i} - 2\mathbf{X}_i\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}_i'\right] \\
&\quad + \sum_{s=1}^{k}(1 + \delta_2 + \cdots + \delta_s)\mathbf{X}_i\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}_s'\mathbf{X}_s\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}_i' \\
&= \left[\mathbf{I}_{n_i} - 2\mathbf{X}_i\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}_i' + \mathbf{X}_i\left(\mathbf{X}'\mathbf{X}\right)^{-1}\sum_{s=1}^{k}\mathbf{X}_s'\mathbf{X}_s\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}_i'\right] \\
&\quad + \left(\sum_{t=2}^{i}\delta_t\right)\left[\mathbf{I}_{n_i} - 2\mathbf{X}_i\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}_i'\right] \\
&\quad + \sum_{s=2}^{k}\left(\sum_{t=2}^{s}\delta_t\right)\mathbf{X}_i\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}_s'\mathbf{X}_s\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}_i' \\
&= \left[\mathbf{I}_{n_i} - \mathbf{X}_i\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}_i'\right] + \left(\sum_{t=2}^{i}\delta_t\right)\left[\mathbf{I}_{n_i} - 2\mathbf{X}_i\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}_i'\right] \\
&\quad + \sum_{t=2}^{k}\sum_{s=t}^{k}\delta_t\mathbf{X}_i\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}_s'\mathbf{X}_s\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}_i' \\
&= \mathbf{D}_{\delta i}.
\end{aligned}$$

Therefore $Z_i' Z_i \overset{D}{=} \sum_{j=1}^{n_i} \lambda_j \left(\mathbf{D}_{\delta i}\right) \mathbf{V}_{ij}$. Since $p_i^2 = Z_i' Z_i / \left(n_i - p\right)$, the lemma is proved.

□

Lemma A.4 *For any symmetric matrix* $\mathbf{A}$ *that is a function of scalar* δ*,*

$$\frac{\partial \lambda_j \left(\mathbf{A}\right)}{\partial \delta} = x_j' \frac{\partial \mathbf{A}}{\partial \delta} x_j$$

where x_j *is the orthonormal eigenvector associated with the* j^{th} *eigenvalue of* $\mathbf{A}$*.*

Proof of Lemma A.4:

Observe that since $\mathbf{A} x_j = \lambda_j \left(\mathbf{A}\right) x_j$, then

$$\begin{aligned} \frac{\partial \mathbf{A} x_j}{\partial \delta} &= \frac{\partial \lambda_j(\mathbf{A}) x_j}{\partial \delta} \\ \Leftrightarrow \quad \mathbf{A} \frac{\partial x_j}{\partial \delta} + \frac{\partial \mathbf{A}}{\partial \delta} x_j &= \lambda_j \left(\mathbf{A}\right) \frac{\partial x_j}{\partial \delta} + \frac{\partial \lambda_j(\mathbf{A})}{\partial \delta} x_j. \end{aligned}$$

Pre-multiplying each side by x_j' yields

$$\begin{aligned} x_j' \mathbf{A} \frac{\partial x_j}{\partial \delta} + x_j' \frac{\partial \mathbf{A}}{\partial \delta} x_j &= x_j' \lambda_j \left(\mathbf{A}\right) \frac{\partial x_j}{\partial \delta} + x_j' \frac{\partial \lambda_j(\mathbf{A})}{\partial \delta} x_j \\ \Leftrightarrow \quad x_j' \lambda_j \left(\mathbf{A}\right) \frac{\partial x_j}{\partial \delta} + x_j' \frac{\partial \mathbf{A}}{\partial \delta} x_j &= x_j' \lambda_j \left(\mathbf{A}\right) \frac{\partial x_j}{\partial \delta} + \frac{\partial \lambda_j(\mathbf{A})}{\partial \delta} x_j' x_j \\ \Leftrightarrow \quad x_j' \frac{\partial \mathbf{A}}{\partial \delta} x_j &= \frac{\partial \lambda_j(\mathbf{A})}{\partial \delta} x_j' x_j. \end{aligned}$$

Since x_j is orthonormal, $x_j' x_j = 1$. Thus the lemma is proved.

□

Proof of Theorem 3.1:

From Lemma A.3 we know that (10) is equivalent to

$$\frac{\sum_{i=1}^{n_k} \lambda_i \left(\mathbf{D}_{\delta k}\right) \mathbf{V}_{ki}}{\sum_{j=1}^{n_1} \lambda_j \left(\mathbf{D}_{\delta 1}\right) \mathbf{V}_{1j}} \geq_{stochastically} \frac{\sum_{i=1}^{n_k} \lambda_i \left(\mathbf{D}_{0k}\right) \mathbf{V}_{ki}}{\sum_{j=1}^{n_1} \lambda_j \left(\mathbf{D}_{01}\right) \mathbf{V}_{1j}} \ . \tag{15}$$

A sufficient condition for (15) (and for (10)) to hold is that

$$\frac{\sum_{i=1}^{n_k} \lambda_i \left(\mathbf{D}_{\delta k}\right) \mathbf{V}_{ki}}{\sum_{j=1}^{n_1} \lambda_j \left(\mathbf{D}_{\delta 1}\right) \mathbf{V}_{1j}} \geq_{a.s.} \frac{\sum_{i=1}^{n_k} \lambda_i \left(\mathbf{D}_{0k}\right) \mathbf{V}_{ki}}{\sum_{j=1}^{n_1} \lambda_j \left(\mathbf{D}_{01}\right) \mathbf{V}_{1j}}, \tag{16}$$

which is equivalent to

$$\left[\sum_{i=1}^{n_k} \lambda_i \left(\mathbf{D}_{\delta k}\right) \mathbf{V}_{ki}\right] \left[\sum_{j=1}^{n_1} \lambda_j \left(\mathbf{D}_{01}\right) \mathbf{V}_{1j}\right] - \left[\sum_{i=1}^{n_k} \lambda_i \left(\mathbf{D}_{0k}\right) \mathbf{V}_{ki}\right] \left[\sum_{j=1}^{n_1} \lambda_j \left(\mathbf{D}_{\delta 1}\right) \mathbf{V}_{1j}\right] \geq_{a.s.} 0$$

$$\Leftrightarrow \quad \sum_{i=1}^{n_k} \sum_{j=1}^{n_1} \left[\lambda_i\left(\mathbf{D}_{\delta k}\right) \lambda_j\left(\mathbf{D}_{01}\right) - \lambda_i\left(\mathbf{D}_{0k}\right) \lambda_j\left(\mathbf{D}_{\delta 1}\right)\right] \mathbf{V}_{ki} \mathbf{V}_{1j} \quad \geq_{a.s.} \quad 0.$$

As $\mathbf{V}_{1j}$ and $\mathbf{V}_{ki}$ are non-negative random variables, the above condition is equivalent to

$$\lambda_i\left(\mathbf{D}_{\delta k}\right) \lambda_j\left(\mathbf{D}_{01}\right) - \lambda_i\left(\mathbf{D}_{0k}\right) \lambda_j\left(\mathbf{D}_{\delta 1}\right) \geq 0, \tag{17}$$

for all $i = 1, \ldots, n_k$ and $j = 1, \ldots, n_1$.

Observe that

$$\mathbf{D}_{\delta 1} = \left[\mathbf{I}_{n_1} - \mathbf{P}_{11}\right] + \sum_{t=2}^{k} \left[\delta_t \sum_{s=t}^{k} \mathbf{P}_{1s} \mathbf{P}_{s1}\right], \text{ and}$$

$$\begin{aligned}
\mathbf{D}_{\delta k} &= \left[\mathbf{I}_{n_k} - \mathbf{P}_{kk}\right] + \left(\sum_{t=2}^{k} \delta_t\right)\left[\mathbf{I}_{n_k} - 2\mathbf{P}_{kk}\right] + \sum_{t=2}^{k}\left[\delta_t \sum_{s=t}^{k} \mathbf{P}_{ks} \mathbf{P}_{sk}\right] \\
&= \left[\mathbf{I}_{n_k} - \mathbf{P}_{kk}\right] + \sum_{t=2}^{k} \delta_t \left[\left(\mathbf{I}_{n_k} - \mathbf{P}_{kk}\right)^2 + \sum_{s=t}^{k-1} \mathbf{P}_{ks} \mathbf{P}_{sk}\right].
\end{aligned}$$

Thus $\mathbf{D}_{\delta k}$ and $\mathbf{D}_{\delta 1}$ may each be written in the form $\mathbf{A} + \delta_t \mathbf{B}$ where $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices and $\mathbf{B}$ is non-negative definite. Then from Lemma A.4, $\partial \lambda_j\left(\mathbf{D}_{\delta i}\right) / \partial \delta_t$ is of the form $x_j' \mathbf{B} x_j \geq 0$. Hence $\lambda_i\left(\mathbf{D}_{\delta k}\right)$ and $\lambda_j\left(\mathbf{D}_{\delta 1}\right)$ are each increasing in δ_t for $t = 2, \ldots, k$.

Also, note that $\lambda_j\left(\mathbf{D}_{01}\right) \geq 0$ and $\lambda_i\left(\mathbf{D}_{0k}\right) \geq 0$ as the eigenvalues of non-negative definite matrices are non-negative. Thus $\lambda_i\left(\mathbf{D}_{\delta k}\right) \lambda_j\left(\mathbf{D}_{01}\right)$ and $\lambda_i\left(\mathbf{D}_{0k}\right) \lambda_j\left(\mathbf{D}_{\delta 1}\right)$ are each increasing in δ_t. Since the left-hand-side of (17) is linear in δ_t and equals 0 when $\delta \equiv 0$, then an equivalent condition for (17) is

$$\frac{\partial \lambda_i\left(\mathbf{D}_{\delta k}\right) \lambda_j\left(\mathbf{D}_{01}\right)}{\partial \delta_t} \geq \frac{\partial \lambda_i\left(\mathbf{D}_{0k}\right) \lambda_j\left(\mathbf{D}_{\delta 1}\right)}{\partial \delta_t}.$$

Since this expression is equivalent to (11), the theorem is proved.

$\square$

Proof of Corollary 3.2:

Since $\mathbf{X}_2'\mathbf{X}_2 = \mathbf{X}'\mathbf{X} - \mathbf{X}_1'\mathbf{X}_1$, it follows that $\mathbf{D}_{\delta 1} = \left[\mathbf{I}_{n_1} - \mathbf{P}_{11}\right] + \delta_2\left(\mathbf{P}_{11} - \mathbf{P}_{11}^2\right)$. From Lemma A.4 $\left(\partial / \partial \delta_2\right) \lambda_j\left(\mathbf{D}_{\delta 1}\right) = u_j'\left(\mathbf{P}_{11} - \mathbf{P}_{11}^2\right) u_j$ where u_j is the orthonormal eigenvector associated with the j^{th} eigenvalue of $\mathbf{D}_{\delta 1}$. Since $\mathbf{D}_{\delta 1}$ is a polynomial in $\mathbf{P}_{11}$, therefore u_j is also an eigenvector of $\mathbf{P}_{11}$ and $\mathbf{P}_{11}^2$, with eigenvalues $\lambda_j\left(\mathbf{P}_{11}\right)$ and $\lambda_j\left(\mathbf{P}_{11}\right)^2$ respectively. Thus

$$\frac{\partial \lambda_j\left(\mathbf{D}_{\delta 1}\right)}{\partial \delta_2} = u_j'\left[\lambda_j\left(\mathbf{P}_{11}\right) - \lambda_j\left(\mathbf{P}_{11}\right)^2\right] u_j = \left[\lambda_j\left(\mathbf{P}_{11}\right) - \lambda_j\left(\mathbf{P}_{11}\right)^2\right].$$

Additionally note that $\mathbf{D}_{\delta 2} = [\mathbf{I}_{n_2} - \mathbf{P}_{22}] + \delta_2 [\mathbf{I}_{n_2} - \mathbf{P}_{22}]^2$. Let v_i be the orthonormal eigenvector associated with the i^{th} eigenvalue of $\mathbf{D}_{\delta 2}$, then by a similar argument as above we have

$$\begin{aligned}
\frac{\partial \lambda_i (\mathbf{D}_{\delta 2})}{\partial \delta_2} &= v_i' \left([\mathbf{I}_{n_2} - \mathbf{P}_{22}]^2 \right) v_i \\
&= v_i' \lambda_i \left([\mathbf{I}_{n_2} - \mathbf{P}_{22}]^2 \right) v_i \\
&= \lambda_i \left([\mathbf{I}_{n_2} - \mathbf{P}_{22}]^2 \right) \\
&= [1 - \lambda_i (\mathbf{P}_{22})]^2 .
\end{aligned}$$

Hence the sufficient condition (11) reduces to verifying

$$\lambda_j (\mathbf{D}_{01}) [1 - \lambda_i (\mathbf{P}_{22})]^2 \geq \lambda_i (\mathbf{D}_{02}) \left[\lambda_j (\mathbf{P}_{11}) - \lambda_j (\mathbf{P}_{11})^2 \right]$$

for all $i = 1, \ldots, n_2$ and $j = 1, \ldots, n_1$. Note that $\lambda_j (\mathbf{D}_{01}) = 1 - \lambda_j (\mathbf{P}_{11})$, and $\lambda_i (\mathbf{D}_{02}) = 1 - \lambda_i (\mathbf{P}_{22})$. Therefore, the above inequality is equivalent to

$$[1 - \lambda_j (\mathbf{P}_{11})] [1 - \lambda_i (\mathbf{P}_{22})]^2 \geq [1 - \lambda_i (\mathbf{P}_{22})] \lambda_j (\mathbf{P}_{11}) [1 - \lambda_j (\mathbf{P}_{11})] \tag{18}$$

for all $i = 1, \ldots, n_2$ and $j = 1, \ldots, n_1$.

Since $\lambda (\mathbf{AB}) = \lambda (\mathbf{BA})$, therefore

$$\begin{aligned}
\lambda (\mathbf{P}_{11}) &= \lambda \left[(\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}_1'\mathbf{X}_1 \right] \\
&= \lambda \left[(\mathbf{X}'\mathbf{X})^{-1} (\mathbf{X}'\mathbf{X} - \mathbf{X}_2'\mathbf{X}_2) \right] \\
&= \lambda \left[\mathbf{I}_p - (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}_2'\mathbf{X}_2 \right] \\
&= 1 - \lambda \left[(\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}_2'\mathbf{X}_2 \right] \\
&= 1 - \lambda (\mathbf{P}_{22}) .
\end{aligned}$$

Hence (18) is equivalent to, for all $i = 1, \ldots, n_2$ and $j = 1, \ldots, n_1$

$$[1 - \lambda_j (\mathbf{P}_{11})] \lambda_i (\mathbf{P}_{11})^2 \geq \lambda_i (\mathbf{P}_{11}) \lambda_j (\mathbf{P}_{11}) [1 - \lambda_j (\mathbf{P}_{11})] . \tag{19}$$

If $\lambda_j (\mathbf{P}_{11}) = 1$ or $\lambda_j (\mathbf{P}_{11}) = 0$, then (19) is trivially satisfied. If $\lambda_j (\mathbf{P}_{11}) \neq 1$ and $\lambda_j (\mathbf{P}_{11}) \neq 0$, then (19) reduces to

$$\lambda_i (\mathbf{P}_{11}) \geq \lambda_j (\mathbf{P}_{11}) \quad \forall i = 1, \ldots, n_2 \text{ and } \; j = 1, \ldots, n_1.$$

Hence all nonzero eigenvalues of $\mathbf{P}_{11}$, which are not equal to 1, must be identical.

$\square$

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